IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

A multicomponent water wave equation

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1985 J. Phys. A: Math. Gen. 18 L1119

(http://iopscience.iop.org/0305-4470/18/18/001)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 129.252.86.83
The article was downloaded on 31/05/2010 at 10:47

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. **18** (1985) L1119–L1122. Printed in Great Britain

LETTER TO THE EDITOR

A multicomponent water wave equation

B A Kupershmidt

The University of Tennessee Space Institute, Tullahoma, Tennessee 37388, USA

Received 14 August 1985

Abstract. A vector extension of the classical dispersive water wave equation is shown to be a member of an infinite integrable three-Hamiltonian hierarchy.

The approach of the category theory is to treat every individual object as a member of a class of related objects. An equally fruitful idea is to consider a given object as a sub- or factor object of a larger system possessing some of the basic properties of the original entity. Besides being an efficient way towards possible generalisations, this procedure allows one to assemble experimental information about systems which could be considered to be similar.

In the theory of integrable systems this approach has been sporadically used for the nonlinear Schrödinger equation (Gibbons 1981) and for the Korteweg-de Vries equation (Hirota and Satsuma 1981, Wilson 1982, Ito 1982), by extending a scalar system into a multicomponent one. In this letter I show that the classical dispersive water wave (DWW) equation (Broer 1975, Kaup 1975) allows a non-trivial vector extension which preserves three-Hamiltonian structures of the original system (Kupershmidt 1985a).

Consider the following system

$$\dot{u}=(2h+u^2-u_x)_x \qquad \dot{h}=(2uh+h_x+\boldsymbol{q}^t\boldsymbol{q}/2)_x \qquad \dot{\boldsymbol{q}}=(u\boldsymbol{q})_x \tag{1}$$

for functions u, h, and $\boldsymbol{q}=(q_1,\ldots,q_N)^t$ of x and t. The time t in (1) is minus twice the fluid dynamical time. For $\boldsymbol{q}=0$ (1) becomes the DWW equation. Denote $\partial=\partial/\partial x$ $(\cdot)^{(k)}=\partial^k(\cdot)$, and let

$$H_0=h \qquad H_1=uh+\boldsymbol{q}^t\boldsymbol{q}/2 \qquad H_2=u^2h+h^2+uh^{(1)}+u\boldsymbol{q}^t\boldsymbol{q}/2 \tag{2}$$

be the first three conservation laws (CL) of the system (1). Then the system (1) can be cast into the following bi-Hamiltonian form

$$(\dot{u},\dot{h},\dot{\boldsymbol{q}}^t)^t=B^1(\delta H_2)=B^2(\delta H_1) \tag{3}$$

where $\delta H=(\delta H/\delta u, \delta H/\delta h, \delta H/\delta\bar{q}^t)^t$ is the vector of variational derivatives of H, and the matrices B^1 and B^2 in (3) produce the following motion equations

$$\dot{u}=\partial(\delta H/\delta h) \qquad \dot{h}=\partial(\delta H/\delta u) \qquad \dot{\boldsymbol{q}}=\partial(\delta H/\delta\bar{q}) \qquad (B^1) \tag{4a}$$

$$\begin{aligned}\dot{u}=2\partial(\delta H/\delta u)+\partial(u-\partial)/(\delta H/\delta h) \qquad \dot{h}&=(u+\partial)\partial(\delta H/\delta h)\\ +(h\partial+\partial h)(\delta H/\delta h)+\boldsymbol{q}^t\partial(\delta H/\delta\bar{q}) \qquad \dot{\boldsymbol{q}}&=\partial(\boldsymbol{q}\delta H/\delta H) \qquad (B^2).\end{aligned} \tag{4b}$$

We see from (4a) that the matrix B^1 is skewsymmetric constant coefficient; thus, B^1 is Hamiltonian (Manin 1979, ch I). The matrix B^2 has the following Lie algebraic interpretation (see Kupershmidt 1985b, ch VIII, § 5): let K be a commutative algebra with a derivation $\partial: K \to K$ (e.g. $K = C^\infty(\mathbb{R}^1)$, $\partial = \mathrm{d}/\mathrm{d}x$); let $D(K)$ be K considered as a Lie algebra with the commutator

$$[X, Y] = XY^{(1)} - X^{(1)}Y \qquad X, Y \in K. \tag{5}$$

Let $\mathcal{Y} = D(K) \ltimes K^{N+1}$ be the semidirect product Lie algebra with the commutator

$$[(f; X; \boldsymbol{r}^t)^t, (g; Y; \boldsymbol{p}^t)^t]^t = (Xg^{(1)} - Yf^{(1)}; XY^{(1)} - X^{(1)}Y; (X\boldsymbol{p}^{(1)} - Y\boldsymbol{r}^{(1)})^t)$$
$$X, Y \in D(K), f, g \in K, \boldsymbol{r}, \boldsymbol{p} \in K^N. \tag{6}$$

Let ω and ν be the following bilinear skewsymmetric forms on $\mathcal{Y}$:

$$\omega[(f; X; \boldsymbol{r}^t)^t, (g; Y; \boldsymbol{p}^t)^t] = 2fg^{(1)} \tag{7}$$

$$\nu[(f; X; \boldsymbol{r}^t)^t, (g; Y; \boldsymbol{p}^t)^t] = -fY^{(2)} + Xg^{(2)}. \tag{8}$$

It is easy to check that ω and ν are generalised two-cocycles on $\mathcal{Y}$. The matrix B^2 in (4b) is the natural Hamiltonian matrix associated with the two-cocycle $\omega + \nu$ on $\mathcal{Y}$.

Let us see that the bi-Hamiltonian definition

$$B^1(\delta H_{n+1}) = B^2(\delta H_n) \tag{9}$$

can be iterated for all n. It will imply that we have the whole hierarchy (9) of bi-Hamiltonian systems. Denote $F_n = \delta H_n = (a_n, b_n, \boldsymbol{c}_n^t)^t$. Writing (9) in a longer form, we obtain

$$b_{n+1}^{(1)} = (2a_n + ub_n - b_n^{(1)})^{(1)} \tag{10a}$$

$$a_{n+1}^{(1)} = ua_n^{(1)} + a_n^{(2)} + h^{(1)}b_n + 2hb_n^{(1)} + \boldsymbol{q}^t\boldsymbol{c}_n^{(1)} \tag{10b}$$

$$\boldsymbol{c}_{n+1}^{(1)} = (\boldsymbol{q}b_n)^{(1)} \tag{10c}$$

so that we indeed can find b_{n+1} and $\boldsymbol{c}_{n+1}$ as

$$b_{n+1} = 2a_n + ub_n - B_n^{(1)} \qquad \boldsymbol{c}_{n+1} = \boldsymbol{q}b_n. \tag{11}$$

To show that the RHS of (10b) belongs to Im∂, notice that the chain of relations

$$F_m^t B^2(F_n) \sim -F_n^t B^2(F_m) = -F_n^t B^1(F_{m+1}) \sim F_{m+1}^t B^1(F_n) = F_{m+1}^t B^2(F_{n-1}) \tag{12}$$

implies

$$F_m^t B^2(F_n) \sim 0 \tag{13}$$

where $a \sim b$ means $(a - b) \in \mathrm{Im}\partial$. In particular, taking $m = 0$ in (13) and using (2), we get

$$ua_n^{(1)} + a_n^{(2)} + h^{(1)}b_n + 2hb_n^{(1)} + \boldsymbol{q}^t\boldsymbol{c}_n^{(1)} = (\delta H_0)^t B^2(\delta H_n) \sim 0$$

so that we indeed can find a_{n+1} from (10b). It remains to show that, for each n, the vector obtained $F_n = (a_n, b_n, \boldsymbol{c}_n^t)^t$ is the vector of variational derivatives of some function H_n. This fact is equivalent to the Fréchet derivative $D(F_n)$ being symmetric (Manin 1979, ch I, Kupershmidt 1980, ch II): $D(F_n)^t = D(F_n)$. Applying the Fréchet derivative operator D to (10) and denoting $D(F_n)$ by D_n, we find that

$$B^1 D_{n+1} = B^2 D_n + G_n \tag{14n}$$

$$G_n := \begin{pmatrix} \partial B_n & 0 & \mathbf{0} \\ a_n^{(1)} & B_n^{(1)} + \partial B_n & \boldsymbol{c}_n^{(1)t} \\ \mathbf{0} & \mathbf{0} & \partial B_n \mathbb{1} \end{pmatrix}. \tag{15}$$

We show that D_n is symmetric by induction on n, the cases $n = 0, 1, 2$ being evidently satisfied. Consider the expression $0 = (14n)B^1 - (14n - 1)B^2$, written as

$$B^1 D_{n+1} B^1 = (B^2 D_n B^1 + B^1 D_n B^2) - B^2 D_{n-1} B^2 + \bar{G}_n \tag{16}$$

$$\bar{G}_n := G_n B^1 - G_{n-1} B^2. \tag{17}$$

Since D_{n+1} is symmetric when $B^1 D_{n+1} B^1$ is, then to make the induction step we need to show, as can be seen from (16), that $\bar{G}_n$ is symmetric, and the latter fact can be checked by a straightforward calculation with the use of the following identities:

$$\{\boldsymbol{c}^{(1)t}\partial \boldsymbol{q} - \boldsymbol{q}^t \boldsymbol{c}^{(1)}\partial\} \quad \text{is symmetric} \tag{18}$$

$$\{a^{(1)}\partial(u - \partial) - (ua^{(1)} + a^{(2)})\partial\} \quad \text{is symmetric} \tag{19}$$

$$\{(b^{(1)} + \partial b)(h\partial + \partial h) - (2hb^{(1)} + h^{(1)}b)\partial\} \quad \text{is symmetric} \tag{20}$$

$$[\partial b \partial (u - \partial)]^t = (b^{(1)} + \partial b)(u + \partial)\partial - [(ub)^{(1)} - b^{(2)}]\partial \tag{21}$$

$$(\partial b \boldsymbol{q}^{(1)t})^t = b^{(1)}\boldsymbol{q}\partial - (\boldsymbol{q}b)^{(1)}\partial. \tag{22}$$

In conclusion, let us see that our bi-Hamiltonian hierarchy (9) is, in fact, a three-Hamiltonian hierarchy

$$B^1(\delta H_{n+1}) = B^2(\delta H_n) = B^3(\delta H_{n-1}) \qquad n \geqslant 0 \tag{23}$$

with $H_{-1} = u/2$. For this, notice than a_n enters only through its derivative $a_n^{(1)}$ in the RHS of (10). Therefore, we can use (11) to iterate (10):

$$\begin{aligned} b_{n+1}^{(1)} &= 2(ua_{n-1}^{(1)} + a_{n-1}^{(2)} + h^{(1)}b^{n-1} + 2hb_{n-1}^{(1)} + \boldsymbol{q}^t \boldsymbol{c}_{n-1}^{(1)}) + \partial(u - \partial)(b_{n-1}) \\ &= 2(u\partial + \partial u)(a_{n-1}) + [2(h\partial + \partial h) + \partial(u - \partial)^2](b_{n-1}) + 2\boldsymbol{q}^t\partial(\boldsymbol{c}_{n-1}) \\ a_{n+1}^{(1)} &= (u + \partial)[(u + \partial)\partial(a_{n-1}) + (h\partial + \partial h)(b_{n-1}) + \boldsymbol{q}^t\partial(\boldsymbol{c}_{n-1})] \\ &\quad + (h\partial + \partial h)[2a_{n-1} + (u - \partial)(b_{n-1})] + \boldsymbol{q}^t\partial\boldsymbol{q}(b_{n-1}) \\ &= [2(h\partial + \partial h) + (u + \partial)^2\partial](a_{n-1}) + [(u + \partial)(h\partial + \partial h) + (h\partial + \partial h)(u - \partial) \\ &\quad + \boldsymbol{q}^t\partial\boldsymbol{q}](b_{n-1}) + (u + \partial)\boldsymbol{q}^t\partial(\boldsymbol{c}_{n-1}) \\ \boldsymbol{c}_{n+1}^{(1)} &= \partial\boldsymbol{q}(2a_{n-1} + ub_{n-1} - b_{n-1}^{(1)}) = 2\partial\boldsymbol{q}(a_{n-1}) + \partial\boldsymbol{q}(u - \partial)(b_{n-1}). \end{aligned} \tag{24}$$

Thus, we see that our matrix B^3 produces the following motion equations:

$$\begin{aligned} \dot{u} &= 2(u\partial + \partial u)(\delta H/\delta u) + [2(h\partial + \partial h) + \partial(u - \partial)^2](\delta H/\delta h) + 2\boldsymbol{q}^t\partial(\delta H/\delta \boldsymbol{q}) \\ \dot{h} &= [2(h\partial + \partial h) + (u + \partial)^2\partial](\delta H/\delta u) + [(u + \partial)(h\partial + \partial h) + (h\partial + \partial h)(u - \partial) \\ &\quad + \boldsymbol{q}^t\partial\boldsymbol{q}](\delta H/\delta h) + (u + \partial)\boldsymbol{q}^t\partial(\delta H/\delta \boldsymbol{q}) \\ \dot{\boldsymbol{q}} &= 2\partial\boldsymbol{q}(\delta H/\delta u) + \partial\boldsymbol{q}(u - \partial)(\delta H/\delta h) \end{aligned} \tag{25}$$

and (23) is obviously satisfied also for $n = 0$. It remains to show that B^3 is a Hamiltonian matrix. The easiest way to do this would be to use a $\boldsymbol{q}$-extended Miura map for the DWW equation from § 3 in Kupershmidt (1985a). However, I was not able to find such a map. The direct check of the Hamiltonian property of the matrix B^3 would be too painful to contemplate. An acceptable way out is this: using the fact that B^3 is Hamiltonian in the absence of $\boldsymbol{q}$, as was proven in § 3 in Kupershmidt (1985a), one has to pay attention only to the new terms due to the presence of $\boldsymbol{q}$. In this way the computation is tolerably tedious. The matrix B^3 is indeed Hamiltonian.

This work was partially supported by the National Science Foundation.

References

Broer L J F 1975 *Appl. Sci. Res.* **31** 377
Gibbons J 1981 *DIAS preprint* STP-81-06 (Dublin)
Hirota R and Satsuma J 1981 *Phys. Lett.* **85A** 407
Ito M 1982 *Phys. Lett.* **91A** 335
Kaup D J 1975 *Prog. Theor. Phys.* **54** 396
Kupershmidt B A 1980 *Lecture Notes Math.* **775** 162
—— 1985a *Commun. Math. Phys.* **98**
—— 1985b *Rev. Asterisque* **123**
Manin Yu I 1979 *J. Sov. Math.* **11** 1
Wilson G 1982 *Phys. Lett.* **89A** 332